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Hyers Ulam Stability of First Order Difference Equations

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Abstract— *In this paper we have investigate the Hyers Ulam stability of first order difference equation of the form* $\Delta y(n) - y(n) = 0$.

1. INTRODUCTION

The study of stability problems for various functional equations originated from a talk given by S.M. Ulam in 1940 [17], Ulam discussed a number of important unsolved problems. Among such problems, a problem concerning the stability of functional equations, “Give conditions in order for a linear mapping nearly an approximately linear mapping to exist” is one of them.

In 1941, Hyer [2] gave an answer to the problem as follows:

Let E_1 and E_2 be two real Banach spaces and $f: E_1 \to E_2$ be a mapping. If there exists an $\in > 0$ such that

$$\left\| f(x+y) - f(x) - f(y) \right\| < \varepsilon, (x, y \in E_1)$$

Then there exists a unique additive mapping $T: E_1 \to E_2$ such that $\left\| f(x) - T(x) \right\| < \varepsilon, x \in E_1$

Furthermore, the results of the Hyers has been generalized by Rassias [14]. After that researchers has extended, the Ulams stability problems to functional equations and generalized Hyer’s result in various directions (see[3, 8, 9, 15]). Thereafter, Ulam’s stability problem for functional equations was replaced by stability of differential/difference equations.

The differential equation,

$$a_n(t)y^n(t) + a_{n-1}(t)y^{n-1}(t) + ... + a_1(t)y^1(t) + a_0(t)y(t) + h(t) = 0$$

has Hyper’s Ulam stability, if for given $\in > 0$, I be an open interval and for any function f satisfying the differential inequality,

$$\left| \begin{matrix} a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + ... + a_1(t)y^1(t) \\ +a_0(t)y(t) + h(t) \end{matrix} \right| \le \varepsilon,$$

then there exists a function f(t) of the above equation such that $\left| f(t) - f_0(t) \right| \le K(\in)$ and $\lim_{\varepsilon \to 0} K(\varepsilon) = 0, t \in I.$

If the proceeding statements is also true when we replace ε and $K(\varepsilon)$ by $\phi(t)$ and $\psi(t)$ respectively, where $\phi, \psi : I \to [0, \infty)$ are functions independent of f and f_0 explicitly, then we say that the corresponding differential equation has the generalized Hyers-Ulam stability or Hyer’s Ulam Rassias stability.

Definition 1: The difference equation

$$a_k(n)y(n+k) + a_{k-1}(n)y(n+k-1) + ... + a_1(n)y(n+1) + a_0(n)y(n) + h(n) = 0$$

has the Hyer’s Ulam stability, if forgiven $\in > 0, I$ be an open interval and for any function f satisfying the inequality,

$$\left| \begin{matrix} a_k(n)y(n+k) + a_{k-1}(n)y(n+k-1) + ... \\ +a_1(n)y(n+1) + a_0(n)y(n) + h(n) \end{matrix} \right| \le \varepsilon$$

then there exists a solution f_0 of the above difference equation such that $\left| f(n) - f_0(n) \right| \le K(\in)$ and $\lim_{\in \to 0} K(\in) = 0$ for $n \in I$.

On Hyer’s Ulam stability, several works have been done in the field of differential equation. Obloza seems to be the 1st author who has investigated the Hyers-Ulam stability of linear differential equation (see for e.g. ([12, 15]). After that Alison and Ger published their work in [1], where they have proved that Hyers-Ulam stability of the differential equation $y^1(t) = y(t)$. On this direction we refer some of the work [4, 5, 6, 7, 10, 17, 13, 16] to the readers and the references cited therein.

The objective of this work is to investigate the Hyers-Ulam stability of the difference equation,

$$\Delta y(n) - y(n) = 0 \qquad (1)$$

where Δ is the forward difference operation, which is defined as $\Delta f(x) = f(x+h) - f(x), h$ is the spacing between two consecutive arguments.

By a solution of (1) we man a real valued function y(n) which satisfies (1).

2. HYERS-ULAM STABILITY OF (1)

This section deals with some useful Lemmas and Hyers-Ulam stability result of (1).

Lemma 1(a): Assume that $Z : ¥ \to ¡$ be a real valued function. The inequality $z(n) \le \Delta z(n)$ is true for all $n \in N$, if and only if there exists a real function $\alpha : ¥ \to ¡$ such that $2^{n+1}\Delta\alpha(n) \ge 0$ and $z(n) = 2^n \alpha(n)$ for all $n \in ¥$.

Proof: Assume that the inequality $z(n) \le \Delta z(n)$ hold for any $n \in ¥$. Let us define a function $\alpha : ¥ \to ¡$ to be

$$\alpha(n) = \left[2^{-n}\right] z(n) \qquad (2)$$

Taking difference operator both sides, we get

$$\Delta\alpha(n) = 2^{-(n+1)} z(n+1) - \left[2^{-n}\right] z(n)$$

$$\Rightarrow 2^{n+1}\Delta\alpha(n) = z(n+1) - 2z(n)$$

$$\Rightarrow 2^{n+1}\Delta\alpha(n) = \Delta z(n) - z(n)$$

As $\quad \Delta z(n) - z(n) \ge 0.$

So, $\quad 2^{n+1}\Delta\alpha(n) \ge 0.$

Also from (2), $z(n) - 2^n \alpha(n)$

Conversely assume that there exists a real function $\alpha : ¥ \to ¡$ such that $2^{n+1}\Delta\alpha(n) \ge 0$ for all $n \in ¥$ and $z(n) : ¥ \to ¡$ by $z(n) = 2^n \alpha(n)$. we have to prove that $z(n) \le \Delta z(n)$.

Now,

$$\Delta z(n) = \Delta\left(2^n \alpha(n)\right)$$

$$= \alpha(n+1)\Delta(n) + 2^n \Delta\alpha(n)$$

$$= \alpha(n+1)\left[2^{n+1} - 2^n\right] + 2^n \Delta\alpha(n)$$

$$= \alpha(n+1)\left[2-1\right] + 2^n \Delta\alpha(n)$$

$$= 2^n \alpha(n+1) - 2^n \alpha(n) + 2^n \alpha(n) + 2^n \Delta\alpha(n)$$

$$= 2^n \alpha(n) + 2^n \Delta\alpha(n) + 2^n \alpha(n)$$

$$= 2^{n+1}\Delta\alpha(n+1) + 2^n \alpha(n)$$

$$= 2^{n+1}\Delta\alpha(n+1) + z(n)$$

$$\Rightarrow \Delta z(n) - z(n) = 2^{n+1}\Delta\alpha(n+1) \ge 0$$

$$\Rightarrow \Delta z(n) \ge z(n)$$

$$\Rightarrow z(n) \le \Delta z(n) \qquad \text{(Hence the Lemma)}$$

Lemma1(b): Assume that $Z : ¥ \to ¡$ be a real valued function. The inequality $z(n) \ge \Delta z(n)$ holds true for any $n \in ¥$ if and only if there exists a real function $\beta : ¥ \to ¡$ such that $2^{n+1}\Delta\beta(n) \le 0$ and $z(n) = 2^n \beta(n)$.

The proof of the lemma is similar to Lemma 1(a), hence omitted.

Theorem 2: Given $\in > 0$. A function $y : ¥ \to ¡$ satisfies inequality

$$\left|\Delta y(n) - y(n)\right| \le \in \quad \forall n \in ¥ \qquad (3)$$

If and only if there exists a function $\alpha : ¥ \to ¡$ such that

$$y(n) = \in + 2^n \alpha(n) \qquad (4)$$

and $\quad 0 \le \Delta\alpha(n) \le 2^{-n} \in \quad \forall n \in ¥ \qquad (5)$

Proof: Assume that y(n) is a real function such that $y : ¥ \to ¡$ is solution of the inequality (3) so

$$\left|\Delta y(n) - y(n)\right| \le \in$$

$$\Rightarrow y(n) - \in \le \Delta y(n) \le y(n) + \in, \forall n \in ¥ \quad (6)$$

Define $z(n) = y(n) - \in$. So from inequality (6), $z(n) \le \Delta y(n)$ holds for any $n \in ¥$. Then by applyingLemma1(a) there exists a real function $\alpha : ¥ \to ¡$ such that

$$2^{n+1}\Delta\alpha(n) \ge 0 \qquad (7)$$

and $$z(n) = 2^n \alpha(n).$$

$$\Rightarrow y(n) - \in = 2^n \alpha(n)$$

$$\Rightarrow y(n) = \in + 2^n \alpha(n)$$

Similarly by defining $z(n) = y(n) + \in$ and using the Hyers of (6) we have $z(n) \ge \Delta y(n)\ \forall n \in ¥$. So according to Lemma 1(b) there exists a real function $\beta : ¥ \to ¡$ such that

$$2^{n+1}\Delta\beta(n) \le 0 \quad (8)$$

and $$z(n) = 2^n \beta(n)$$

$$\left.\begin{aligned} &\Rightarrow y(n) + \in = 2^n \beta(n) \\ &\Rightarrow y(n) = -\in + 2^n \beta(n) \end{aligned}\right. \qquad (9)$$

Applying difference operator for both the equations (4) and(9)we get,

$$\Delta y(n) = \Delta\left(\in + 2^n \alpha(n)\right)$$
$$\Rightarrow \Delta y(n) = \alpha(n+1)2^{n+1} - \alpha(n)2^n$$

Again $$\begin{aligned} \Delta y(n) &= \beta(n+1)2^{n+1} - 2^n \beta(n) \\ &= 2^n\left(2\beta(n+1) - \beta(n)\right) \end{aligned}$$

$$= 2^n\left[\Delta\beta(n) + \beta(n+1)\right] = 2^n \Delta\beta(n) + 2^n \beta(n+1)$$

$$\Rightarrow \Delta\beta(n) = 2^{-n}\Delta y(n) - \beta(n+1)$$

$$= 2^n\left[y(n+1) - y(n)\right] - \left(y(n+1) + \in\right)\left[2^{-(n+1)}\right]$$

$$\Rightarrow \Delta\beta(n) = \Delta\alpha(n) - 2 \in 2^{-(n-1))}$$

$$\Rightarrow 2^{n+1}\Delta\beta(n+1) = 2^{n+1}\Delta\alpha(n) - 2\in$$

As $$2^{n+1}\Delta\beta(n) \le 0 \Rightarrow 2^{n+1}\Delta\alpha(n) - 2\in \le 0$$

$$\Rightarrow 2\in \ge 2^{n+1}\Delta\alpha(n),$$

$$\Rightarrow 0 \le 2^{n+1}\Delta\alpha(n) \le 2\in$$

$$\Rightarrow 0 \le \Delta\alpha(n) \le 2^{-n} \in,\ \forall n \in ¥.$$

Conversely,

Assume that a function $y : ¥ \to ¡$ is given by (4) $\forall n \in ¥$, where $\alpha : ¥ \to ¡$ is a real function satisfies (5) for any $n \in ¥$.

Now, $\Delta y(n) - y(n) = y(n+1) - 2y(n)$

$$= \in + \alpha(n+1)2^{n+1} - 2(\in + \alpha(n)2^n)$$

$$= 2^{n+1}\Delta\alpha(n) - \in$$

$$\le 2\in - \in = \in$$

So $$\Delta y(n) - y(n) \le \in \qquad (10)$$

Similarly from (9), $y(n) = -\in + \beta(n)2^n$

Now, $\Delta y(n) - y(n)$

$$= y(n+1) - 2y(n)$$

$$= \in + \beta(n+1)2^{n+1} + 2\in - \beta(n)2^{n+1}$$

$$= 2^{n+1}\Delta\beta(n) + \in$$

$$\ge -2\in \beta \in = -\in$$

So $$\Delta yn) - y(n) \ge -\in \qquad (11)$$

Combining (10) & (11), we get

$$-\in \le \Delta y(n) - y(n) \le \in \quad \forall n \in ¥.$$

$$\Rightarrow \left|\Delta y(n) - y(n)\right| \le \in$$

Hyer-Ulam Stability of difference equation (1) i.e., $\Delta y(n) - y(n) = 0$.

Theorem 3: If a function $y : ¥ \to ¡$ satisfies equality (3) i.e. $\left|\Delta y(n) - y(n)\right| \le \in$ for $n \in ¥$, then there exists a real no, C such that $\left|y(n) - C2^n\right| \le \in \forall n \in ¥$.

Proof: Given $\in > 0$. Since y satisfies (3) then by theorem 2, there exists a function $\alpha : ¥ \to ¡$ such that $y(n) = \in + 2^n \alpha(n)$ and $0 \le \Delta\alpha(n) \le 2^{n-2} \in,\ n \in ¥$.

Define $C = \lim_{k\to\infty} \alpha(k)$

$$0 \le \Delta\alpha(n) \le 2^{-n} \in \Rightarrow -2^{-n} \in \le -\Delta\alpha(n) \le 0$$

$$\Rightarrow -\in \sum_{j=n}^{k-1} 2^{-j} \le -\sum_{j=n}^{k-1} \Delta\alpha(j) \le 0,\ k > n$$

But $\sum_{j=n}^{k-1} \Delta\alpha(j) = \sum_{j=n}^{k-1} \left(\alpha(j+1) - \alpha(j)\right) = \alpha(k) - \alpha(n)$

and $\sum_{j=n}^{k-1} 2^{-j} = \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots + \frac{1}{2^{k-1}} = \frac{1}{2^{n-1}} - \frac{1}{2^{k-1}}$

Hence $-\in \left(\frac{1}{2^{n-1}} - \frac{1}{2^{k-1}}\right) \le \alpha(n) - \alpha(k) \le 0$

Taking limit as $k \to \infty$, we get $\frac{-\in}{2^{n-1}} \le \alpha(n) - \alpha(k) \le 0$

$$\Rightarrow \frac{-\in 2^n}{2^{n-1}} \le 2^n \alpha(n) - 2^n C \le 0$$

$$\Rightarrow -2 \in \le 2^n \alpha(n) - 2^n C \le 0$$

$$\Rightarrow -\in \le \in +2^n \alpha(n) - 2^n C \le \in$$

$\Rightarrow -\in \le y(n) - 2^n C \le \in \qquad \Rightarrow \left|y(n) - C2^n\right| \le \in, n \in$ ¥.

(Hence the Theorem)

REFERENCE

[1] C.Alishan, R.Ger: On some inequalities and stability results related to the exponential function,J.Inequal.Appl, 2(1998)373-380.

[2] D. H. Hyers: On the stability of the linear functional equations, Proc.Natl.Acad.Sci.,USA. 27(1941)222-234.

[3] K. W Jun, Y. H .Lee: A generalization of the Hyers Ulam Rassias stability of the Jensons equation, J. Math. Anal.Appl, 238 (1999),305-315.

[4] S. M. Jung: Hyers Ulam stability of linear differential equations of first order,Appl. Math. Lett., 17 (2004),1135-1140.

[5] S. M. Jung: Hyers Ulam stability of linear differential equations of first Order (III), J. Math. Ana. And Appl.,311(2005),139-146.

[6] S. M. Jung: Hyers Ulam stability of linear differential equations of first Order (II), Appl. Math. Lett., 19 (2006),854-858.

[7] S. M. Jung, Th. M. Rassias: Ulam's proble for approximate homomorphisms in connection with Bernoulli differential equation, Appl. Math. Comp., 187 (2007),223-227.

[8] T. Miura, S.E.Takahasi, H. Choda: On the Hyers Stability of real continuous function valued differentiable map, Tokyp J. Math (2001),467-476.

[9] T.Miura: On the Hyers Ulam stability of differentiable map, Sci. Math. Japan,55 (2002), 17-24

[10] T. Miura, S. miyajima, S.E, Takahasi: A characterisation of Hyers Ulam stability of first Order linear differential operators, J.Math,Anal.Appl., 286(2003),136-146.

[11] T. Miura, S. M. Jung, S.E, Takahasi: Hyers Ulam stability of the banach space valued linear differential equations y '=ay, Korean Math.Soc,41 (2004),995-1005

[12] M. Obloza: Hyers stability of the linear differential equation, Rocznik Nauk-Dydakt, proce.Mat.,13 (1993)259-270.

[13] M. Obloza: Connection between Hyers and Lyapunov stability of the ordinary differential Equations, Rocznik Nauk-Dydakt, proce. Mat.,14 (1997) 141-146

[14] Th. M. Rassias: on the stability of linear mapping in Banach spaces, Proc. Amer.Math. Soc., 72(1978),297-300.

[15] Th. M. Rassias: on the stability of functional equations and aproblem of Ulam, Acta .Appl. Math., 62(2000) 23-20

[16] S. E Takahasi, H.Takagi, T.Miura, S. Miyajima: The Hyers ulam stability constants of first order linear differential operators, J. Math.Anal.Appl., 296 (2004),403-409.

[17] S. M. Ulam: Problems in Modern Mathematics,Chapter-VI, Wiely, New York (1964)